Some Peculiarities of Newton-Hooke Space-Times

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Abstract

Newton-Hooke space-times are the non-relativistic limit of (anti-)de Sitter space-times. We investigate some peculiar facts about the Newton-Hooke space-times, among which the "extraordinary Newton-Hooke quantum mechanics" and the "anomalous Newton-Hooke space-times" are discussed in detail. Analysis on the Lagrangian/action formalism is performed in the discussion of the Newton-Hooke quantum mechanics, where the path integral point of view plays an important role, and the physically measurable density of probability is clarified.

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I. INTRODUCTION

The Newton-Hooke (NH) space-times [1] are very interesting objects in theoretical physics. On one hand, they are non-relativistic space-times, which have an absolute time and are relatively simple. On the other hand, they have non-vanishing curvature and are akin to the constant curvature space-times, de Sitter (dS) space-time and anti-de Sitter (AdS) space-time, in many aspects. Due to their non-relativistic nature, the standard non-relativistic quantum mechanics can be established on them, which has been detailedly discussed in [2] and will be called the ordinary NH quantum mechanics.[1] However, due to their non-flat (and even non-static) nature, there are interesting subtle points in the NH quantum mechanics, which is intimately related to the Schrödinger group [3], as the "conformal" extension of the NH groups (see e.g. [4]), and the Hermiticity of the NH Hamiltonian. In particular, the NH quantum mechanics is only established in [2] by invariance considerations directly on the Schrödinger equation, while it is not clear whether we can obtain this equation by some (canonical or path-integral) quantization procedure. We give thorough clarification of these points in this paper, starting from analysis on the Lagrangian (or action) formalism for the NH dynamics and making use of a systematic path integral point of view. Moreover, it is pointed out that the probability density defined in [2] is not NH-invariant and so cannot be the physically measurable one, while the observable density of probability in NH space-times is proposed. As a byproduct, an elegant geometric interpretation of the harmonic oscillator/free particle correspondence [5] will be described.

The standard NH space-times as affine connection spaces have also been discussed in [2], with the affine connection uniquely determined by the NH invariance of both the connection itself and the affine parameter. In fact, it is taken for granted that the affine parameter is identified, up to constant linear transformations, to the NH-invariant proper time τ. Interestingly, it turns out that consistent mechanics can be established on affine connection spaces with NH-invariant connections and a non-invariant affine parameter, which will be called the anomalous NH space-times. The so-called linear coordinates on the (anomalous) NH space-times are naturally introduced there, under which the NH "isometry" transformations become linear. It is also interesting to see that these coordinates are related to

[1] Although [2] basically discusses the case of space dimension $d = 3$, it is trivial to generalize most of the discussions to the case of arbitrary d.

the Schrödinger group. Furthermore, as an example of non-trivial dynamics, the Newton-Cartan-like gravity [2] in the anomalous NH space-times will be shown in this paper.

This paper is organized as follows. We first briefly review the NH space-times and NH mechanics in Sec.II, concentrating on the symmetry and quantum aspects. Then the subtleties in the NH quantum mechanics are analyzed in Sec.III, where the so-called extraordinary NH quantum mechanics are proposed. The anomalous NH space-times are discussed in Sec.IV finally.

II. NEWTON-HOOKE SPACE-TIMES AND NEWTON-HOOKE MECHANICS

First, we review some relevant facts about the NH space-times and the mechanics on them. The NH space-times can be described from either the algebraic or the geometric point of view.

Algebraically, the $(1+d)$-dimensional NH space-times can be constructed as the homogeneous spaces $N_\pm(1,d)/\hat{N}_\pm(1,d)$, where $N_\pm(1,d)$ are the NH groups with positive/negative "cosmological constant" and $\hat{N}_\pm(1,d)$ the corresponding homogeneous NH groups, with the upper/lower sign sometimes called the NH/anti-NH (ANH) case, respectively. The NH groups are certain non-relativistic limit (contraction [6]) of the dS/AdS groups. The Lie algebras of the NH groups, called $\mathfrak{n}_\pm(1,d)$, are described by the following Lie brackets between anti-Hermitian generators:

$$
\begin{aligned}
[J_{ij}, J_{kl}] &= \delta_{jk}J_{il} + \delta_{il}J_{jk} - \delta_{ik}J_{jl} - \delta_{jl}J_{ik}, \\
[J_{ij}, P_k] &= \delta_{jk}P_i - \delta_{ik}P_j, \qquad [J_{ij}, K_k] = \delta_{jk}K_i - \delta_{ik}K_j, \\
[H, P_j] &= \pm\nu^2 K_j, \qquad [H, K_j] = P_j, \qquad [P_i, K_j] = 0, \\
[J_j, H] &= 0, \qquad i,j,k = 1,\cdots,d,
\end{aligned} \tag{1}
$$

where ν, sometimes called the NH constant, is the characteristic of the NH groups (and the corresponding NH space-times). Here, as usual, J_{jk} are the generators of rotation, P_j that of space translation, K_j that of (NH) boosts, and H that of time translation. When $\nu \to 0$, the NH groups just become the Galilei group. The homogeneous-space structure $N_\pm(1,d)/\hat{N}_\pm(1,d)$ of the NH space-times can be used to systematically obtain variant quantities on them, including the so-called "canonical" connection [7, 8], from the viewpoint of which the NH space-times can be regarded as affine connection spaces.

Geometrically, the NH space-times can be directly obtained by the non-relativistic limit of the dS/AdS space-times, which gives more straightforwardly the coordinates on the NH space-times and the explicit NH group actions on them (the so-called NH transformations) as a whole, thanks to the simple pseudo-sphere nature of the dS/AdS space-times. Here we omit the limiting procedure and only list some relevant results.[2] The NH space-times are of topology $\mathbb{R}^{d+1}$. In order to explicitly describe their geometry, some suitable coordinate systems are needed. Two coordinate systems are frequently used in the literature. One is the so-called Beltrami coordinates (t, x^i) (with $-\nu^{-1} < t < \nu^{-1}$ in the NH case), which is the non-relativistic limit of the Beltrami coordinates on the dS/AdS space-times [10]. The other is the so-called static coordinates (τ, q^i), which is the non-relativistic limit of the well-known static coordinates on the dS/AdS space-times. The relation between these two coordinate systems is

$$\begin{aligned} \tau &= \begin{cases} \nu^{-1} \tanh^{-1} \nu t & \text{(for NH)} \\ \nu^{-1} \tan^{-1} \nu t & \text{(for ANH)} \end{cases}, \qquad (2) \\ q^i &= \frac{x^i}{\sigma(t)^{1/2}}, \qquad \sigma(t) \equiv 1 \mp \nu^2 t^2, \end{aligned}$$

where τ is just the NH-invariant proper time (periodic in the ANH case). The NH transformations under the Beltrami coordinates take a simple, fractional linear form:

$$t' = \frac{t - a^t}{\sigma(a^t, t)}, \qquad \sigma(a^t, t) \equiv 1 \mp \nu^2 a^t t, \tag{3}$$

$$x'^i = \frac{\sigma(a^t)^{1/2}}{\sigma(a^t, t)} O^i{}_j (x^j - a^j - u^j t), \tag{4}$$

where $(O^i{}_j) \in SO(d)$ and $a^t, a^j, u^j \in \mathbb{R}$ are parameters of space rotation, time translation, space translation and boost, respectively, so these coordinates are more suitable in this context. It should be emphasized that the transformation for t is independent of x^i, so the time simultaneity is absolute, similar to the Newtonian case. Moreover, the (degenerate) metric

$$d\tau^2 = \sigma^{-2}(t) dt^2 \tag{5}$$

is invariant under NH transformations. The generators in (1) are realized under the Beltrami

[2] For detailed discussions of the limiting procedure, see [2, 9].

coordinates as[3]

$$H=\sigma(t)\partial_t\mp\nu^2 tx^i\partial_i,\qquad P_i=-\partial_i,\qquad K_i=-t\partial_i,\qquad J_{ij}=x^i\partial_j-x^j\partial_i.$$

Taking the Beltrami coordinate systems as inertial (reference) frames, the NH transformations then act as transformations among inertial frames, corresponding to the Galilei transformations in Newtonian mechanics. Differentiation of the NH transformation (3,4) gives rise to the velocity composition law [2] and the following transformation of (3-)acceleration:

$$\frac{dv'^i}{dt'}=\frac{\sigma^3(a^t,t)}{\sigma^{3/2}(a^t)}O^i{}_j\frac{dv^j}{dt},\tag{6}$$

with $v^i\equiv dx^i/dt$ the velocity. The homogeneous form of the above transformation means that uniform-velocity motions transform among themselves under change of inertial frames, implying the existence of the NH first law of the same form as the Newton's first law. In fact, it has been shown early that a free point particle moves along a straight line with uniform velocity in the NH space-time [9]. Moreover, the NH second law of the same form as the Newton's second law also exists provided the force F^i has the same NH transformations as the acceleration. In fact, the NH third law is also consistent with the NH transformation, and is really respected for the gravitational interaction on the NH space-times [2].

The NH algebra has a unique central extension $\mathfrak{n}_\pm^{\mathrm{C}}(1,d)$:

$$[P_i,K_j]=-i\delta_{ij}m,\qquad \text{and the other Lie brackets same as } \mathfrak{n}_\pm(1,d),\tag{7}$$

in generic space dimension d, with m a central element.[4] The Schrödinger equation on the NH space-times can be deduced by the algebraic construction from the extended NH algebra (7) or by the geometric contraction from the Klein-Gordon equation on the dS/AdS space-times [2]. Both methods lead to an NH-invariant equation

$$\mathrm{i}\hbar\partial_t\psi(t,x)=[-\frac{\hbar^2\nabla^2}{2m}\pm\frac{\mathrm{i}\hbar\nu^2 tx^i\partial_i}{\sigma(t)}\mp\frac{m\nu^2x^2}{2\sigma^2(t)}]\psi(t,x),\tag{8}$$

where we have ignored possible NH-invariant interaction terms. The conservation of probability for this equation and its relation to the NH fluid mechanics has been shown [2], giving some justification of the probability interpretation to be physical.

[3] Summation of repeated indices is assumed throughout this paper, unless otherwise indicated.
[4] The case of $d=2$ is very special, where there exists the so-called exotic central extension. See [11, 12].

III. EXTRAORDINARY NEWTON-HOOKE QUANTUM MECHANICS

We base our discussions on the Lagrangian formalism of the NH mechanics. It is not difficult to show that the non-relativistic (NH) limit [9] of the action functional

$$S = -\int mc^2 d\tau$$

for a free point particle in the dS/AdS space-times is

$$S[x(t)] = \int \frac{1}{2}\frac{m}{\sigma(t)}[\dot{x}^2 \mp \nu^2 (x - t\dot{x})^2 \pm 2\frac{\nu^2 x^2}{\sigma(t)}]dt \tag{9}$$

under the Beltrami coordinates, where an over dot means $\frac{d}{dt}$. Note that in the process of taking this limit, we have not discarded any boundary terms. The above action functional implies a time-dependent Lagrangian

$$L = \frac{1}{2}\frac{m}{\sigma(t)}[\dot{x}^2 \mp \nu^2 (x - t\dot{x})^2 \pm 2\frac{\nu^2 x^2}{\sigma(t)}] \tag{10}$$

for the NH mechanics, which leads to the following canonical momenta and Hamiltonian:

$$\begin{aligned} p_i &= \frac{m}{\sigma(t)}[\dot{x}^i \pm \nu^2 t(x^i - t\dot{x}^i)] = m[\dot{x}^i \pm \frac{\nu^2 t x^i}{\sigma(t)}], \\ H &= p\cdot\dot{x} - L = \frac{1}{2}\frac{m}{\sigma(t)}[\dot{x}^2 \pm \nu^2(x^2 - t^2\dot{x}^2) \mp 2\frac{\nu^2 x^2}{\sigma(t)}] \\ &= \frac{1}{2}m[(\frac{p}{m} \mp \frac{\nu^2 t x}{\sigma(t)})^2 \mp \frac{\nu^2(1 \pm \nu^2 t^2)x^2}{\sigma(t)^2}] \\ &= \frac{p^2}{2m} \mp \frac{\nu^2 t x\cdot p}{\sigma(t)} \mp \frac{m\nu^2 x^2}{2\sigma(t)^2}. \end{aligned} \tag{11}$$

Roughly, the NH Schrödinger equation (8) can be obtained from the canonical quantization of the above Hamiltonian, while we will come back to the subtle problem of operator ordering later. The corresponding canonical equations of motion are

$$\begin{aligned} \dot{x}^i &= \{x^i, H\} = \frac{p_i}{m} \mp \frac{\nu^2 t x^i}{\sigma(t)}, \\ \dot{p}_i &= \{p_i, H\} = \pm\frac{\nu^2 t p_i}{\sigma(t)} \pm \frac{m\nu^2 x^i}{\sigma(t)^2}, \end{aligned}$$

which clearly leads to

$$\ddot{x}^i = \frac{\dot{p}_i}{m} \mp \frac{d}{dt}[\frac{\nu^2 t}{\sigma(t)}]x^i \mp \frac{\nu^2 t}{\sigma(t)}\dot{x}^i = 0$$

after a bit of calculation. The same EOM $\ddot{x}^i = 0$ can, of course, be obtained directly as the Euler-Lagrange equation from the Lagrangian (10).

Having the above Lagrangian formalism, it is easy to discuss the NH quantum mechanics, especially the realization of the NH symmetry in the quantum case from a path integral point of view. Formally, the propagator (Feynman kernel) from the space-time point (t_1, x_1) to (t_2, x_2) is given by the path integral

$$K(t_2, x_2; t_1, x_1) = \int e^{\frac{\mathrm{i}}{\hbar}S[x(t)]} D_{t_1,x_1}^{t_2,x_2} x(t),$$

where $D_{t_1,x_1}^{t_2,x_2} x(t)$ means the appropriate functional integral measure for functions $x(t)$ satisfying $x(t_1) = x_1$ and $x(t_2) = x_2$. Take the NH space translation

$$\begin{aligned} t' &= t, \\ x'^i &= x^i - a^i \end{aligned} \tag{12}$$

as an example. This transformation acts nontrivially on $S[x(t)]$ as

$$\begin{aligned} S[x(t)] \to S'[x'(t')] &= \int_{t_1}^{t_2} \frac{1}{2}\frac{m}{\sigma(t)}[\dot{x}'^2 \mp \nu^2(x' - t\dot{x}')^2 \pm 2\frac{\nu^2 x'^2}{\sigma(t)}]dt \\ &= \int_{t_1}^{t_2} \frac{1}{2}\frac{m}{\sigma(t)}[\dot{x}^2 \mp \nu^2(x - a - t\dot{x})^2 \pm 2\frac{\nu^2 (x-a)^2}{\sigma(t)}]dt \\ &= S[x(t)] + \int_{t_1}^{t_2} \frac{1}{2}\frac{m}{\sigma(t)}[\pm 2\nu^2 a\cdot(x - t\dot{x}) \mp \nu^2 a^2 \mp 4\frac{\nu^2 a\cdot x}{\sigma(t)} \pm 2\frac{\nu^2 a^2}{\sigma(t)}]dt \\ &= S[x(t)] + \int_{t_1}^{t_2} d\left(\mp\frac{m\nu^2 t a\cdot x}{\sigma(t)} \pm \frac{m\nu^2 t a^2}{2\sigma(t)}\right) \\ &= S[x(t)] + \left(\mp\frac{m\nu^2 t a\cdot x}{\sigma(t)} \pm \frac{m\nu^2 t a^2}{2\sigma(t)}\right)\Bigg|_{t_1}^{t_2}, \end{aligned}$$

which leads to

$$\begin{aligned} &K(t_2, x_2; t_1, x_1) \to \\ &K'(t'_2, x'_2; t'_1, x'_1) = \int e^{\frac{\mathrm{i}}{\hbar}S'[x'(t')]} D_{t_1,x'_1}^{t_2,x'_2} x'(t') \\ = \ &\exp\frac{\mathrm{i}}{\hbar}\left[\left(\mp\frac{m\nu^2 t_2 a\cdot x_2}{\sigma(t_2)} \pm \frac{m\nu^2 t_2 a^2}{2\sigma(t_2)}\right) - \left(\mp\frac{m\nu^2 t_1 a\cdot x_1}{\sigma(t_1)} \pm \frac{m\nu^2 t_1 a^2}{2\sigma(t_1)}\right)\right] K(t_2, x_2; t_1, x_1). \end{aligned} \tag{13}$$

Combined with the basic property

$$\psi(t_2, x_2) = \int K(t_2, x_2; t_1, x_1)\psi(t_1, x_1) d^d x_1 \tag{14}$$

of the propagator, the NH transformation (13) immediately gives the transformed wave function

$$\psi'(t', x') = \exp\frac{\mathrm{i}}{\hbar}\left(\mp\frac{m\nu^2 t a\cdot x}{\sigma(t)} \pm \frac{m\nu^2 t a^2}{2\sigma(t)}\right)\psi(t, x)$$

under the NH space translation, which is the same as in [2]. Other kinds of NH transformations on $\psi(t,x)$ can be similarly obtained, with the same results as in [2], but in a more systematic and straightforward way. Note that for the NH space translation (12), the path-integral measure in (13), as well as the measure $d^d x_1$ in (14), is obviously invariant, but for the NH time translation there is possibly a real factor from the nontrivial transformation of the path-integral measure, which cannot be obtained in the above simple discussion. Nevertheless, the full NH time translation of $\psi(t,x)$, and so the nontrivial transformation of the path-integral measure, can be determined by the NH invariance of the Schrödinger equation (8), as has been done in [2]. Considering the corresponding infinitesimal transformations, it is easy to check that this quantum system carries a realization of the extended NH algebra (7).

The EOM $\ddot{x}^i = 0$ is the same as that of a free point particle in the Galilei space-time, which can be given by the simple Lagrangian

$$L = \frac{1}{2} m \dot{x}^2, \tag{15}$$

so it may be expected that the complicated Lagrangian (10) is equivalent to the above simple one, with the understanding that the latter is an expression under the Beltrami coordinates in the NH space-times. In fact, that is the case. We have

$$\frac{1}{2}\frac{m}{\sigma(t)}[\dot{x}^2 \mp \nu^2(x - t\dot{x})^2 \pm 2\frac{\nu^2 x^2}{\sigma(t)}]dt = \frac{1}{2} m\dot{x}^2 dt + d\left(\pm\frac{m\nu^2 t x^2}{2\sigma(t)}\right), \tag{16}$$

so these two forms of Lagrangian are equivalent up to total derivatives. According to the above discussion from the path integral point of view, discarding this boundary term corresponds to a unitary transformation

$$\psi(t,x) = \exp\frac{\mathrm{i}}{\hbar}\left(\pm\frac{m\nu^2 t x^2}{2\sigma(t)}\right)\tilde{\psi}(t,x) \tag{17}$$

of the wave function. On the one hand, canonical quantization of the simple Lagrangian (15) leads to the standard Schrödinger equation

$$\mathrm{i}\hbar\partial_t\psi(t,x) = -\frac{\hbar^2\nabla^2}{2m}\psi(t,x) \tag{18}$$

for a free point particle, without any ambiguity. On the other hand, a direct transformation (17) of the wave function in (8) gives

$$\mathrm{i}\hbar\partial_t\tilde{\psi}(t,x) = \left(-\frac{\hbar^2\nabla^2}{2m} \mp \frac{\mathrm{i}\hbar\nu^2 t d}{2\sigma(t)}\right)\tilde{\psi}(t,x),$$

which can be transformed into the standard one (18) through

$$\tilde{\psi}(t,x) = \sigma(t)^{d/4}\psi(t,x).$$

Here recall that d is the space dimension. The above transformation is, however, not unitary, so it is nontrivial in physics. In fact, if we take the usual symmetrization prescription for the ordering of x^i and p_i to render Hermiticity of the canonical quantization of the Hamiltonian (11) in the usual sense, we will obtain the Schrödinger equation

$$\mathrm{i}\hbar\partial_t\psi(t,x) = [-\frac{\hbar^2\nabla^2}{2m} \pm \frac{\mathrm{i}\hbar\nu^2 t x^i\partial_i}{\sigma(t)} \pm \frac{\mathrm{i}\hbar\nu^2 t d}{2\sigma(t)} \mp \frac{m\nu^2 x^2}{2\sigma^2(t)}]\psi(t,x),$$

which becomes the same form as (18) but with ψ replaced by $\tilde{\psi}$:

$$\mathrm{i}\hbar\partial_t\tilde{\psi}(t,x) = -\frac{\hbar^2\nabla^2}{2m}\tilde{\psi}(t,x) \tag{19}$$

upon the unitary transformation (17).

Those two Schrödinger equations (8) and (19) are not trivially equivalent in physics, and the former has been discussed in detail in [2] with its NH invariance and conservation of probability shown, which justifies it as the correct equation for the NH quantum mechanics, so we call the quantum mechanics described by the latter the extraordinary NH quantum mechanics. Nevertheless, if we allow non-unitary transformations of the wave function, these two equations are essentially equivalent. In fact, collecting the results of the above discussion, we see the following transformation

$$\psi(t,x) = \sigma(t)^{d/4}\exp\frac{\mathrm{i}}{\hbar}\left(\pm\frac{m\nu^2 t x^2}{2\sigma(t)}\right)\tilde{\psi}(t,x) \tag{20}$$

of wave functions in these two equations. Having this explicit form of transformation, the NH invariance of (19) is evident, with the NH transformation properties listed as follows:

- NH space translation $x'^i = x^i - a^i$: $\tilde{\psi}'(t',x') = \tilde{\psi}(t,x)$.
- NH time translation $t' = \frac{t-a^t}{\sigma(a^t,t)}$: $\tilde{\psi}'(t',x') = \frac{\sigma(a^t,t)^{d/2}}{\sigma(a^t)^{d/4}}\exp\frac{\mathrm{i}}{\hbar}\left(\pm\frac{m\nu^2 a^t x^2}{2\sigma(a^t,t)}\right)\tilde{\psi}(t,x)$.[5]
- NH boost $x'^i = x^i - u^i t$: $\tilde{\psi}'(t',x') = \exp\frac{\mathrm{i}}{\hbar}(-mu\cdot x + \frac{1}{2}mu^2 t)\tilde{\psi}(t,x)$.
- NH space rotation $x'^i = O^i{}_j x^j$: $\tilde{\psi}'(t',x') = \tilde{\psi}(t,x)$.

[5] See Appendix A for details.

The NH space translation, boost and space rotation of the wave function has exactly the same form as the Galilei ones, which can also be easily obtained from the path integral point of view for the simple Lagrangian (15).

Note that the density of probability in the (ordinary) NH quantum mechanics is defined as [2]

$$\rho = \sigma(t)^{-d/2}\psi^*\psi,$$

and then, accordingly, the inner product should be defined as[6]

$$\langle\psi,\phi\rangle = \int \psi^*\phi\sigma(t)^{-d/2}d^dx.$$

So, given the transformation (20), the corresponding expressions for $\tilde{\psi}$ are

$$\rho = \tilde{\psi}^*\tilde{\psi}, \qquad \left\langle\tilde{\psi},\tilde{\phi}\right\rangle = \int \tilde{\psi}^*\tilde{\phi}d^dx,$$

which is consistent with the usual Hermiticity of the canonical quantization of the Hamiltonian (11) mentioned above. These expressions and the standard form

$$j_i = \frac{\mathrm{i}\hbar}{2m}(\tilde{\psi}\partial_i\tilde{\psi}^* - \tilde{\psi}^*\partial_i\tilde{\psi})$$

of the flux of probability can also be directly obtained from the extraordinary NH Schrödinger equation (19). Although ρ and j_i satisfy a form of “conservation of probability" [2]

$$\partial_t\rho + \partial_i j_i = 0 \tag{21}$$

similar to that in flat space-times, however, the probability density ρ (as well as the norm of the inner product defined above) cannot be the physically measurable one, at least in the standard framework of general relativity, since it is not NH-invariant. The standard, observable probability density should be

$$\tilde{\rho} = \psi^*\psi = \sigma(t)^{d/2}\tilde{\psi}^*\tilde{\psi}, \tag{22}$$

and the NH-invariant inner product is

$$\{,\} = \int \psi^*\phi d^dx = \int \tilde{\psi}^*\tilde{\phi}\sigma(t)^{d/2}d^dx. \tag{23}$$

[6] The generator $\hat{H} = \mathrm{i}\sigma(t)\partial_t \mp \mathrm{i}\nu^2 t x^i\partial_i = \mathrm{i}\partial_\tau$ of the NH time translation is Hermitian under this inner product.

It has long been known that the standard Schrödinger equation (18) or (19) for a free point particle is in fact invariant under a larger symmetry group than the Galilei group, called the Schrödinger group [3]. The additional types of transformations in the Schrödinger group are the "dilatation"

$$\begin{aligned} t' &= D^2 t, \\ x'^i &= D x^i, \end{aligned} \tag{24}$$

$$\psi'(t',x') = D^{-d/2}\psi(t,x), \tag{25}$$

and the "special conformal transformation" (SCT)

$$\begin{aligned} t' &= \frac{t}{1-kt}, \\ x'^i &= \frac{x^i}{1-kt}, \end{aligned} \tag{26}$$

$$\psi'(t',x') = (1-kt)^{d/2}\exp\frac{\mathrm{i}}{\hbar}\left(\frac{mkx^2}{2(1-kt)}\right)\psi(t,x), \tag{27}$$

where the unitary factor $\exp\frac{\mathrm{i}}{\hbar}\left(\frac{mkx^2}{2(1-kt)}\right)$ can also be easily read off from the path integral point of view for the Lagrangian (15). In fact, it can be shown that the NH groups are subgroups of the Schrödinger group. For example, an NH time translation can be obtained as the combination of an SCT, a (Galilei) time translation and a dilatation. This situation is just like that the dS, AdS and Poincaré groups are different subgroups of the conformal group [13].

The above discussions also enlighten the study of the harmonic oscillator. It is known that under the static coordinates (2) the NH Schrödinger equation [2, 12]

$$\mathrm{i}\hbar\partial_\tau\psi(\tau,q) = (-\frac{\hbar^2\nabla_q^2}{2m} \mp \frac{1}{2}m\nu^2 q^2)\psi(\tau,q) \tag{28}$$

takes the form of the d-dimensional anti-harmonic/harmonic oscillator with an angular frequency ν, which can also be deduced from the following coordinate transformation:

$$L = \frac{1}{2}\frac{m}{\sigma(t)}[\dot{x}^2 \mp \nu^2(x-t\dot{x})^2 \pm 2\frac{\nu^2 x^2}{\sigma(t)}] = \frac{1}{2}m(\dot{q}^2 \pm \nu^2 q^2)$$

of the Lagrangian. Note that in the above transformation we do not discard any total derivatives and $\psi(\tau,q) = \psi(t,x)$. From (20), we see that the wave-function transformation that relates the (anti-)harmonic oscillator (28) to the free point particle (19) is

$$\psi(\tau,q) = \begin{cases} \mathrm{sech}^{d/2}(\nu\tau)\exp\frac{\mathrm{i}}{\hbar}\left(\frac{1}{2}m\nu\tanh(\nu\tau)q^2\right)\tilde{\psi}(t,x) & \text{(for NH)} \\ \sec^{d/2}(\nu\tau)\exp\frac{\mathrm{i}}{\hbar}\left(-\frac{1}{2}m\nu\tan(\nu\tau)q^2\right)\tilde{\psi}(t,x) & \text{(for ANH)} \end{cases} \tag{29}$$

in terms of the static coordinates. So each solution of the equation (19) gives a solution of the equation (28), and vice versa. This fact has been known for some time [5], but here we deduce it in an elegant geometric way, which makes its meaning very intuitive. Let us present two examples. The first example is the plane wave solution of (19), where

$$\tilde{\psi}(t,x) = e^{\frac{\mathrm{i}}{\hbar}(p\cdot x - \frac{p^2}{2m}t)} = \begin{cases} e^{\frac{\mathrm{i}}{\hbar}[p\cdot q\mathrm{sech}(\nu\tau) - \frac{p^2}{2m\nu}\tanh(\nu\tau)]} & \text{(for NH)} \\ e^{\frac{\mathrm{i}}{\hbar}[p\cdot q\sec(\nu\tau) - \frac{p^2}{2m\nu}\tan(\nu\tau)]} & \text{(for ANH)} \end{cases}$$

gives

$$\psi(\tau,q) = \begin{cases} \mathrm{sech}^{d/2}(\nu\tau)\exp\frac{\mathrm{i}}{\hbar}\left(p\cdot q\mathrm{sech}(\nu\tau) - (\frac{p^2}{2m\nu} - \frac{1}{2}m\nu q^2)\tanh(\nu\tau)\right) & \text{(for NH)} \\ \sec^{d/2}(\nu\tau)\exp\frac{\mathrm{i}}{\hbar}\left(p\cdot q\sec(\nu\tau) - (\frac{p^2}{2m\nu} + \frac{1}{2}m\nu q^2)\tan(\nu\tau)\right) & \text{(for ANH)} \end{cases}.$$

It is straightforward to check that these wave functions do satisfy (28).[7] In fact, it is interesting to notice that in the ANH case the wave function $\psi(\tau,q)$ is of the same form as the propagator of the harmonic oscillator, which is a solution of (28) by definition. The second example is the ground state solution of (28) for the ANH case, where

$$\psi(\tau,q) = e^{-\frac{m\nu}{2\hbar}q^2 - \frac{\mathrm{i}d}{2}\nu\tau} = e^{-\frac{m\nu x^2}{2\hbar\sigma(t)} - \frac{\mathrm{i}d}{2}\tan^{-1}\nu t}$$

gives

$$\tilde{\psi}(t,x) = \sigma(t)^{-d/4}e^{-\frac{\mathrm{i}d}{2}\tan^{-1}\nu t}\exp\left(\frac{\mathrm{i}}{\hbar}\frac{m\nu^2 t x^2}{2\sigma(t)} - \frac{m\nu x^2}{2\hbar\sigma(t)}\right) = (1+\mathrm{i}\nu t)^{-d/2}\exp\left(-\frac{m\nu x^2}{2\hbar(1+\mathrm{i}\nu t)}\right).$$

It is again interesting to notice that this wave function is of essentially the same form as the propagator of the free point particle. Thus, an interesting duality between the harmonic oscillator and the free point particle has been shown, which can be investigated in more general cases.

It is worthwhile to discuss the ANH case, corresponding to the well-behaved harmonic oscillator, in a little more detail. The most important (Hermitian) generators are

$$\hat{H} = \mathrm{i}\hbar\partial_\tau, \qquad \hat{P}_i = -\mathrm{i}\hbar\cos(\nu\tau)\frac{\partial}{\partial q^i}, \qquad \hat{K}_i = -\mathrm{i}\nu^{-1}\hbar\sin(\nu\tau)\frac{\partial}{\partial q^i}$$

[7] It is obvious that even the wave function in the ANH case is not normalizable, so it is not for a bound state.

in terms of the static coordinates[8], which satisfy

$$[\hat{H}, \hat{P}_i] = -\mathrm{i}\hbar\nu^2 \hat{K}_i, \qquad [\hat{H}, \hat{K}_i] = \mathrm{i}\hbar \hat{P}_i.$$

In fact, defining the combinations

$$A_i = \hat{P}_i + \mathrm{i}\nu\hat{K}_i = -\mathrm{i}e^{\mathrm{i}\nu\tau}\frac{\partial}{\partial q^i}, \qquad A_i^\dagger = \hat{P}_i - \mathrm{i}\nu\hat{K}_i = -\mathrm{i}e^{-\mathrm{i}\nu\tau}\frac{\partial}{\partial q^i},$$

we have

$$[\hat{H}, A_i] = -\hbar\nu A_i, \qquad [\hat{H}, A_i^\dagger] = \hbar\nu A_i^\dagger,$$

which clearly shows that A_i and $A_i^\dagger$ act as the lowering and raising operators, respectively, changing the energies of stationary states by one level $\hbar\nu$. Thus, the NH group $N_-(1,d)$ acts as something like the dynamical symmetry group of the harmonic oscillator.

Moreover, since the Schrödinger equation of the free point particle is invariant under the Schrödinger group, having the above duality between the (anti-)harmonic oscillator and the free point particle, we see that the Schrödinger equation (28) of the (anti-)harmonic oscillator is, actually, invariant under the Schrödinger group. There seems no simple expressions for the finite dilatation and SCT under the coordinates (τ, q^i), so we consider the infinitesimal version instead. The generator of the dilatation (denoted by ∂_D) is of the form

$$2t\partial_t + x^i\partial_i = \begin{cases} \nu^{-1}\sinh(2\nu\tau)\partial_\tau + \cosh(2\nu\tau)q^i\frac{\partial}{\partial q^i} & \text{(for NH)} \\ \nu^{-1}\sin(2\nu\tau)\partial_\tau + \cos(2\nu\tau)q^i\frac{\partial}{\partial q^i} & \text{(for ANH)} \end{cases}$$

in terms of (τ, q^i). The commutator of $H = \partial_\tau$ with ∂_D is $[\partial_\tau, \partial_D] = 2\partial_G$, where

$$\partial_G = \begin{cases} \cosh(2\nu\tau)\partial_\tau + \nu\sinh(2\nu\tau)q^i\frac{\partial}{\partial q^i} & \text{(for NH)} \\ \cos(2\nu\tau)\partial_\tau - \nu\sin(2\nu\tau)q^i\frac{\partial}{\partial q^i} & \text{(for ANH)} \end{cases}$$

is actually the combination $2\partial_t - \partial_\tau = (1 \pm \nu^2 t^2)\partial_t \pm \nu^2 t x^i \partial_i$ of the Galilei time translation and the NH time translation[9]. It is easy to check that $[\partial_\tau, \partial_G] = \pm 2\nu^2\partial_D$ and $[\partial_G, \partial_D] = 2\partial_\tau$, so we see that these generators form an $\mathfrak{so}(1,2)$ subalgebra, as required by the structure of

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[8] The centrally extended version is

$$\hat{P}_i = -\mathrm{i}\hbar\cos(\nu\tau)\frac{\partial}{\partial q^i} + m\nu x^i\sin(\nu\tau), \qquad \hat{K}_i = -\mathrm{i}\nu^{-1}\hbar\sin(\nu\tau)\frac{\partial}{\partial q^i} - m x^i\cos(\nu\tau).$$

[9] Such kinds of combinations have been discussed in other context [14].

the Schrödinger group. Supplemented with the generators ∂_D and ∂_G, the extended NH algebra $\mathfrak{n}^{\mathrm{C}}_{\pm}(1,d)$ becomes the full Schrödinger algebra. Now we should work out the actions of these generators on the wave function $\psi(\tau,q)$ using (29), with some important results shown as follows:

- ∂_τ: $\delta\psi = 0$.
- ∂_D: $\delta\psi = \begin{cases} \epsilon(-\frac{d}{2}\cosh 2\nu\tau + \frac{\mathrm{i}}{\hbar}m\nu q^2 \sinh 2\nu\tau)\psi & \text{(for NH)} \\ \epsilon(-\frac{d}{2}\cos 2\nu\tau - \frac{\mathrm{i}}{\hbar}m\nu q^2 \sin 2\nu\tau)\psi & \text{(for ANH)} \end{cases}$ with ϵ the infinitesimal parameter.
- ∂_G: $\delta\psi = \begin{cases} \epsilon(-\frac{d}{2}\nu\sinh 2\nu\tau + \frac{\mathrm{i}}{\hbar}m\nu^2 q^2 \cosh 2\nu\tau)\psi & \text{(for NH)} \\ \epsilon(\frac{d}{2}\nu\sin 2\nu\tau - \frac{\mathrm{i}}{\hbar}m\nu^2 q^2 \cos 2\nu\tau)\psi & \text{(for ANH)} \end{cases}$.

It is a little lengthy but straightforward to check the invariance of (28) under these infinitesimal transformations.

IV. ANOMALOUS NEWTON-HOOKE SPACE-TIMES

The standard NH space-times as affine connection spaces have been discussed in [2], with the following nonzero coefficients of the affine connection:

$$\Gamma^t_{tt} = \frac{\pm 2\nu^2 t}{\sigma(t)}, \qquad \Gamma^i_{tj} = \Gamma^i_{jt} = \frac{\pm\nu^2 t}{\sigma(t)}\delta^i_j.$$

These coefficients cannot be uniquely determined by the NH-invariant metrics, since the latter are degenerate. But they can be uniquely determined by the NH invariance of both the connection itself and the affine parameter, as shown in the Appendix A of [2]. In fact, it is taken for granted in [2] that the affine parameter is identified, up to constant linear transformations, to the NH-invariant proper time τ. However, this identification is too restrictive, since the affine parameter λ is not an observable in general and need not be NH-invariant. Relaxing this restriction,[10] there can be one arbitrary real parameter C,

[10] For doing this, we actually violate the compatibility between the affine connection and the (degenerate) metric, as can be checked for the connection (30), but it seems that no essential problem will be caused in a theory which is not generally covariant.

appearing as the integration constant when solving the first order ODE imposing the NH invariance, in the nonzero coefficients:

$$\Gamma^t_{tt} = \frac{\pm 2\nu^2 t + 2C\nu}{\sigma(t)}, \qquad \Gamma^i_{tj} = \Gamma^i_{jt} = \frac{1}{2}\Gamma^t_{tt}\delta^i_j = \frac{\pm\nu^2 t + C\nu}{\sigma(t)}\delta^i_j. \tag{30}$$

Now the first integral of the temporal component of the geodesic equation:

$$\frac{d^2 t}{d\lambda^2} + \Gamma^t_{\mu\nu}(t,x)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0$$

is (omitting the multiplicative integration constant)

$$\frac{dt}{d\lambda} = \sigma(t)\varsigma(t)^C$$

with

$$\varsigma(t) = \begin{cases} \frac{1-\nu t}{1+\nu t} & \text{(for NH)} \\ e^{-2\tan^{-1}\nu t} & \text{(for ANH)} \end{cases},$$

which means

$$\frac{d\tau}{d\lambda} = \varsigma(t)^C,$$

and further (omitting the additive integration constant)

$$\lambda = \frac{1}{2C\nu}\varsigma(t)^{-C} - \frac{1}{2C\nu}.$$

We should check whether the NH first law is always respected for arbitrary C. In fact, it is not difficult to prove that the integral of the full geodesic equation

$$\frac{d^2 x^\rho}{d\lambda^2} + \Gamma^\rho_{\mu\nu}(t,x)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0$$

is a straight (world) line for arbitrary C. Furthermore, we should check that λ remains affine parameter under NH transformations, i.e. λ transforms linearly, which can be shown as follows:

$$\frac{d\tau}{d\lambda'} = \varsigma(t')^C = \begin{cases} \left(\frac{1-\nu^2 a^t t - \nu t + \nu a^t}{1-\nu^2 a^t t + \nu t - \nu a^t}\right)^C = \left(\frac{1+\nu a^t}{1-\nu a^t}\right)^C \left(\frac{1-\nu t}{1+\nu t}\right)^C & \text{(for NH)} \\ e^{-2C(\tan^{-1}\nu t - \tan^{-1}\nu a^t)} = e^{2C\tan^{-1}\nu a^t} e^{-2C\tan^{-1}\nu t} & \text{(for ANH)} \end{cases} = \frac{\varsigma(t)^C}{\varsigma(a^t)^C}$$

leading to

$$\frac{d\lambda'}{d\lambda} = \varsigma(a^t)^C.$$

Thus the space-times with the NH-invariant metric $d\tau^2$ and affine connection (30) should be considered as qualified NH space-times, which we call the anomalous NH space-times.

It is straightforward to show that the NH-invariant curvature tensor and Ricci tensor corresponding to the connection (30) have the following nonzero components

$$R^i_{t\mu\nu} = \pm\frac{(1\mp C^2)\nu^2}{\sigma(t)^2}(\delta^t_\mu\delta^i_\nu - \delta^i_\mu\delta^t_\nu), \qquad R_{tt} = \mp\frac{(1\mp C^2)\nu^2 d}{\sigma(t)^2}. \tag{31}$$

It can be seen that the case of $C = 1$ for NH is very special[11], where the curvature tensor vanishes and the space-time becomes totally flat. In this case, (30) becomes

$$\Gamma^t_{tt} = \frac{2\nu}{1-\nu t}, \qquad \Gamma^i_{tj} = \Gamma^i_{jt} = \frac{\nu}{1-\nu t}\delta^i_j.$$

Since the space-time is flat, there should be a coordinate system in which the affine connection also vanishes. It can be checked that

$$\begin{aligned} \lambda &= \frac{1}{2\nu}\frac{1+\nu t}{1-\nu t} - \frac{1}{2\nu} = \frac{t}{1-\nu t}, \\ y^i &= \frac{x^i}{1-\nu t} \end{aligned} \tag{32}$$

is just such a coordinate system (with $-(2\nu)^{-1} < \lambda$). The coordinates (λ, y^i) are related to (t, x^i) by a fractional linear transformation (with common denominator), so a free point particle still moves along a straight line with uniform velocity in terms of (λ, y^i). Even if C is arbitrary, (λ, y^i) are interesting and useful coordinates on the NH space-times. In fact, the realization of the NH transformation under these coordinates can be worked out as

$$\begin{aligned} \lambda' &= \frac{t'}{1-\nu t'} = \frac{t-a^t}{1-\nu^2 a^t t - \nu t + \nu a^t} = \varsigma(a^t)\lambda - \frac{a^t}{1+\nu a^t}, \\ y'^i &= \frac{x'^i}{1-\nu t'} = \frac{\sigma(a^t)^{1/2} O^i{}_j(x^j - a^j - u^j t)}{1-\nu^2 a^t t - \nu t + \nu a^t} = \varsigma(a^t)^{1/2} O^i{}_j(y^j - a^j - w^j\lambda) \end{aligned} \tag{33}$$

with $w^j = u^j + \nu a^j$, which is a linear transformation instead of a fractional linear one. Thus we call (λ, y^i) the linear coordinates. Note that (32) is of the same form as the SCT (26), while (33) is of the same form as the dilatation (24) if discarding the inhomogeneous terms, which shows an interesting relationship to the Schrödinger group. The metric (5) under the linear coordinates is

$$d\tau^2 = (1+2\nu\lambda)^{-2} d\lambda^2.$$

It is also possible to consider the Newton-Cartan-like gravity in the anomalous NH space-times, similar to the standard case in [2]. Rewriting the NH second law

$$m\frac{d^2x^i}{dt^2} = F^i$$

[11] The discussion for $C = -1$ is similar.

for the gravitational interaction as

$$\frac{d^2x^i}{d\lambda^2} - \frac{F^i}{m}\frac{dt}{d\lambda}\frac{dt}{d\lambda} + 2\Gamma^i_{tj}\frac{dt}{d\lambda}\frac{dx^j}{d\lambda} = 0$$

with Γ^i_{tj} given in (30), we see that the Newton-Cartan-like connection should be taken as

$$\Gamma^i_{tt} = -\frac{F^i}{m}$$

in addition to (30). The above equation is NH invariant due to the tensor-like property of Γ^i_{tt} under NH transformations, since the same discussion as in the Appendix C of [2] is not spoilt by the introduction of the parameter C. Note that the nonzero components of the curvature tensor and Ricci tensor (31) now become

$$R^i_{ttj} = -R^i_{tjt} = -\partial_j\Gamma^i_{tt} \pm \frac{(1 \mp C^2)\nu^2}{\sigma(t)^2}\delta^i_j, \qquad R_{tt} = \partial_i\Gamma^i_{tt} \mp \frac{(1 \mp C^2)\nu^2 d}{\sigma(t)^2}, \tag{34}$$

before we consider the gravitational field equation. Taking into account the natural constraints in [2], including the NH invariance, we can assume the following form of the field equation:

$$R_{tt} = [4\pi G\rho(t,x) \mp (1 \mp C^2)\nu^2 d]g_{tt},$$

where $g_{tt} = \sigma(t)^{-2}$ is the tt component of the (degenerate) metric, G the gravitational constant, and the mass density $\rho(t,x)$ a scalar field under NH transformations. Together with (34), the above field equation gives

$$\partial_i\Gamma^i_{tt} = \frac{4\pi G\rho(t,x)}{\sigma(t)^2}, \tag{35}$$

so the subsequent discussions are actually the same as in [2], which gives the following NH law of gravity:

$$\frac{d^2x^i}{dt^2} = -\frac{GM}{\sigma(t)^{1/2}}\frac{x^i - X^i}{|x-X|^3} \tag{36}$$

for the case of $d = 3$ and the test particle in the gravitational field of a point-like source with mass M and position $X(t)$. It is not strange that the parameter C does not play an essential role in the Newton-Cartan-like gravity, since it is the NH invariance, not the geometry (affine connection and curvature), that largely determines the final equation (36).

In fact, the anomalous structure extends to the Galilei case. Now the relevant limit is $\nu \to 0$ and $C \to \infty$ but $\gamma \equiv \nu C$ held fixed, which leads to the following nonzero coefficients of the Galilei-invariant affine connection:

$$\Gamma^t_{tt} = 2\gamma, \qquad \Gamma^i_{tj} = \Gamma^i_{jt} = \gamma\delta^i_j$$

and the following nonzero components of the Galilei-invariant curvature tensor and Ricci tensor

$$R^i_{t\mu\nu} = -\gamma^2(\delta^t_\mu\delta^i_\nu - \delta^i_\mu\delta^t_\nu), \qquad R_{tt} = \gamma^2 d.$$

The integral of the temporal component of the geodesic equation is ($\tau = t$ in this case)

$$\frac{dt}{d\lambda} = e^{-2\gamma t}, \qquad \lambda = \frac{1}{2\gamma}e^{2\gamma t},$$

and it is easy to show that the integral of the full geodesic equation is a straight (world) line for arbitrary γ. However, similar to the NH case, since the dynamical equations (for example the Newton's law of gravity) are largely determined by the Galilei invariance, it is not clear whether the parameter γ can play a role in physical observations.

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Appendix A: Newton-Hooke Time Translation of $\tilde{\psi}(t,x)$

We know [2] that the wave function $\psi(t,x)$ is invariant under the NH time translation (3), i.e.

$$\psi'(t',x') = \psi(t,x). \tag{A1}$$

Under the primed reference frame, the wave-function transformation (20) is written as

$$\psi'(t',x') = \sigma(t')^{d/4} \exp\frac{\mathrm{i}}{\hbar}\left(\pm\frac{m\nu^2 t' x'^2}{2\sigma(t')}\right)\tilde{\psi}'(t',x').$$

Now noticing

$$\sigma(t') = 1 \mp \nu^2\frac{(t-a^t)^2}{\sigma(a^t,t)^2} = \frac{\sigma(a^t)\sigma(t)}{\sigma(a^t,t)^2}$$

and that x^i transforms as

$$x'^i = \frac{\sigma(a^t)^{1/2}}{\sigma(a^t,t)}x^i$$

under the NH time translation, we have

$$\begin{aligned}\psi'(t',x') &= \frac{\sigma(a^t)^{d/4}\sigma(t)^{d/4}}{\sigma(a^t,t)^{d/2}}\exp\frac{\mathrm{i}}{\hbar}\left(\pm\frac{m\nu^2(t-a^t)x^2}{2\sigma(t)\sigma(a^t,t)}\right)\tilde{\psi}'(t',x')\\ &= \frac{\sigma(a^t)^{d/4}\sigma(t)^{d/4}}{\sigma(a^t,t)^{d/2}}\exp\frac{\mathrm{i}}{\hbar}\left(\pm\frac{m\nu^2 t x^2}{2\sigma(t)}\mp\frac{m\nu^2 a^t x^2}{\sigma(a^t,t)}\right)\tilde{\psi}'(t',x').\end{aligned}$$

Comparing the above equation with (20) and taking into account (A1), we then obtain the NH time translation of $\tilde{\psi}(t,x)$ to be proved.

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